Shop 'til You Drop!

Are you just browsing? Take your time.
Researchers know, to the dollar, the average amount the typical consumer spends at the shopping mall. And the longer you stay, the more you spend. So if you say you're just browsing, that's just fine with the mall merchants. Browsing is time and, as we will explore in this section, time is money.

Objective #1: Evaluate exponential functions.

✔ Solved Problem #1

1. The exponential function $f(x) = 42.2(1.56)^x$ models the average amount spent, $f(x)$, in dollars, at a shopping mall after x hours. What is the average amount spent, to the nearest dollar, after three hours?

$f(x) = 42.2(1.56)^x$

$f(3) = 42.2(1.56)^3$

≈ 160.20876

≈ 160

The average amount spent after three hours at a mall is $160.

✎ Pencil Problem #1 ✎

1. The exponential function $f(x) = 574(1.026)^x$ models the population of India, $f(x)$, in millions, x years after 1974. Find India's population, to the nearest million, in the year 2028.

Objective #2: Graph exponential functions.

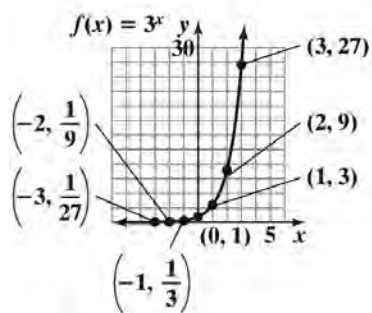### ✔ Solved Problem #2

2a. Graph: $f(x) = 3^x$

Make a table of values:

x	$f(x) = 3^x$	(x, y)
-3	$3^{-3} = \dfrac{1}{27}$	$\left(-3, \dfrac{1}{27}\right)$
-2	$3^{-2} = \dfrac{1}{9}$	$\left(-2, \dfrac{1}{9}\right)$
-1	$3^{-1} = \dfrac{1}{3}$	$\left(-1, \dfrac{1}{3}\right)$
0	$3^0 = 1$	$(0, 1)$
1	$3^1 = 3$	$(1, 3)$
2	$3^2 = 9$	$(2, 9)$
3	$3^3 = 27$	$(3, 27)$

✎ Pencil Problem #2 ✎

2a. Graph: $f(x) = 4^x$

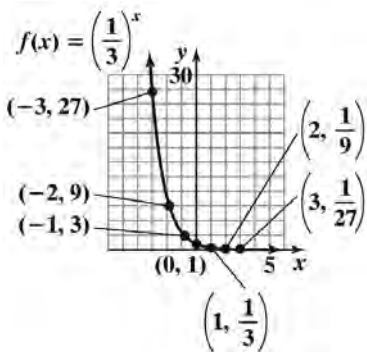Plot the points in the table and connect with a smooth curve.

2b. Graph: $f(x) = \left(\dfrac{1}{3}\right)^x$

Make a table of values:

| x	$f(x) = \left(\dfrac{1}{3}\right)^x$	(x, y)
−3	$\left(\dfrac{1}{3}\right)^{-3} = 27$	$(-3,\ 27)$
−2	$\left(\dfrac{1}{3}\right)^{-2} = 9$	$(-2,\ 9)$
−1	$\left(\dfrac{1}{3}\right)^{-1} = 3$	$(-1,\ 3)$
0	$\left(\dfrac{1}{3}\right)^{0} = 1$	$(0,\ 1)$
1	$\left(\dfrac{1}{3}\right)^{1} = \dfrac{1}{3}$	$\left(1, \dfrac{1}{3}\right)$
2	$\left(\dfrac{1}{3}\right)^{2} = \dfrac{1}{9}$	$\left(2, \dfrac{1}{9}\right)$
3	$\left(\dfrac{1}{3}\right)^{3} = \dfrac{1}{27}$	$\left(3, \dfrac{1}{27}\right)$

Plot the points in the table and connect with a smooth curve.

2b. Graph: $g(x) = \left(\dfrac{3}{2}\right)^x$

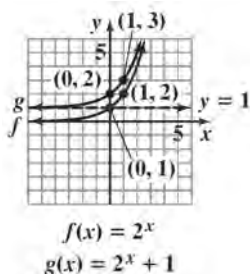**2c.** Use the graph of $f(x) = 3^x$ to obtain the graph of $g(x) = 3^{x-1}$.

The graph of $g(x) = 3^{x-1}$ is the graph of

$f(x) = 3^x$ shifted 1 unit to the right. We identified two points on the graph of *f*, which we graphed in Solved Problem 2a, and added 1 to each of the *x*-coordinates.

$f(x) = 3^x$
$g(x) = 3^{x-1}$
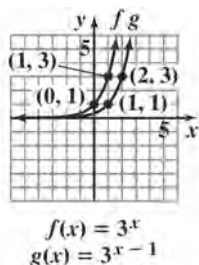
2c. Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^{x+1}$.

2d. Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^x + 1$.

The graph of $g(x) = 2^x + 1$ is the graph of

$f(x) = 2^x$ shifted up 1 unit. We identified two points on the graph of *f* and added 1 to each of the *y*-coordinates. The asymptote is also shifted up 1 unit.

$f(x) = 2^x$
$g(x) = 2^x + 1$

2d. Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^x - 1$.

Objective #3: Evaluate functions with base *e*.

✔ **Solved Problem #3**

3. The exponential function $f(x) = 1144e^{0.0325x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, *x* years after 1978. If trends continue, project the gray wolf's population in the recovery area in 2014.

2014 is 36 years after 1978.

$f(x) = 1144e^{0.0325x}$
$f(36) = 1144e^{0.0325(36)} \approx 3686$

In 2014 the gray wolf population of the Western Great Lakes was projected to be about 3686.

✎ **Pencil Problem #3**

3. The exponential function $g(x) = 32.7e^{0.0217x}$ models the percentage of high school seniors who applied to more than three colleges *x* years after 1980. What percentage of high school seniors applied to more than three colleges in 2016?

| **Objective #4:** Use compound interest formulas.	
✔ *Solved Problem #4*	*Pencil Problem #4*
4a. A sum of \$10,000 is invested at an annual rate of 1.08%. Find the balance in the account after 5 years subject to quarterly compounding. $A = P\left(1+\dfrac{r}{n}\right)^{nt}$ $A = 10{,}000\left(1+\dfrac{0.0108}{4}\right)^{4(5)}$ $\approx \$10{,}554.08$	**4a.** A sum of \$10,000 is invested at an annual rate of 1.32%. Find the balance in the account after 5 years subject to monthly compounding.
4b. A sum of \$10,000 is invested at an annual rate of 1.08%. Find the balance in the account after 5 years subject to continuous compounding. $A = Pe^{rt}$ $A = 10{,}000e^{0.0108(5)}$ $\approx \$10{,}554.85$	**4b.** A sum of \$10,000 is invested at an annual rate of 1.32%. Find the balance in the account after 5 years subject to continuous compounding.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:
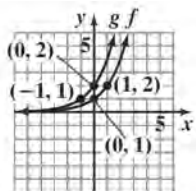
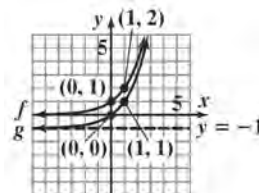**1.** 2295 million *(4.1 #65c)*
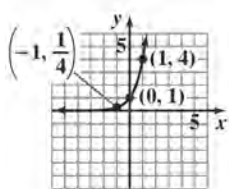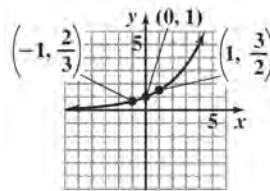
2a. $f(x) = 4^x$ *(4.1 #11)* **2b.** $g(x) = \left(\dfrac{3}{2}\right)^x$ *(4.1 #13)*

2c. $f(x) = 2^x$, $g(x) = 2^x + 1$ *(4.1 #25)* **2d.** $f(x) = 2^x$, $g(x) = 2^x - 1$ *(4.1 #27)*

3. 71% *(4.1 #71b)*

4a. \$10,682.24 *(4.1 #53c)* **4b.** \$10,682.27 *(4.1 #53d)*

Copyright © 2022 Pearson Education Inc.

Speak Up!

The loudness level of a sound is measured in decibels.
Decibel levels range from 0, a barely audible sound,
to 160, a sound resulting in a ruptured eardrum.

We will see that decibels can be modeled by a
logarithmic function, the topic of this section.

Objective #1: Change from logarithmic to exponential form.

✔ *Solved Problem #1*

1. Write the equation $2 = \log_b 25$ in its equivalent
exponential form.

$2 = \log_b 25$

$b^2 = 25$

✎ *Pencil Problem #1* ✎

1. Write the equation $\log_6 216 = y$ in its equivalent
exponential form.

Objective #2: Change from exponential to logarithmic form.

✔ *Solved Problem #2*

2. Write the equation $2^5 = x$ in its equivalent
logarithmic form.

$2^5 = x$

$5 = \log_2 x$

✎ *Pencil Problem #2* ✎

2. Write the equation $7^y = 200$ in its equivalent
logarithmic form.

Objective #3: Evaluate logarithms.

✔ *Solved Problem #3*

3a. Evaluate: $\log_5 \dfrac{1}{125}$.

$\log_5 \dfrac{1}{125} = -3$ because

$5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$

✎ *Pencil Problem #3* ✎

3a. Evaluate: $\log_5 \dfrac{1}{5}$.

3b. Evaluate: $\log_{36} 6$.

3b. Evaluate: $\log_4 16$.

$\log_{36} 6 = \dfrac{1}{2}$ because $36^{\frac{1}{2}} = \sqrt{36} = 6.$

Objective #4: Use basic logarithmic properties.

✔ *Solved Problem #4*

✎ *Pencil Problem #4*✎

4. Evaluate $\log_7 7^8$.

4. Evaluate $\log_4 1$.

Because $\log_b b^x = x,$ we conclude $\log_7 7^8 = 8.$

Objective #5: Graph logarithmic functions.

✔ *Solved Problem #5*

✎ *Pencil Problem #5*✎

5. Graph $f(x) = 3^x$ and $g(x) = \log_3 x$ in the same rectangular coordinate system.
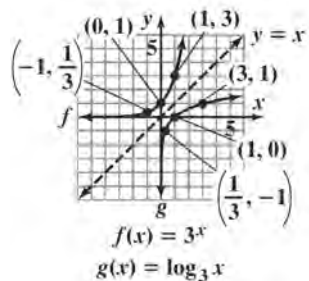
5. Graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \log_{\frac{1}{2}} x$ in the same rectangular coordinate system.

Set up a table of coordinates for $f(x) = 3^x.$

x	-2	-1	0	1	2	3
$f(x) = 3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

Reverse these coordinates to obtain the coordinates of $g(x) = \log_3 x.$

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$g(x) = \log_3 x$	-2	-1	0	1	2	3

$f(x) = 3^x$
$g(x) = \log_3 x$

Objective #6: Find the domain of a logarithmic function.

✔ **Solved Problem #6**	✎ **Pencil Problem #6** ✎
6. Find the domain of $f(x) = \log_4(x-5)$.	**6.** Find the domain of $f(x) = \log(2-x)$.

$x - 5 > 0$

$\quad x > 5$

The domain of f is $(5, \infty)$.

Objective #7: Use common logarithms.

✔ **Solved Problem #7**	✎ **Pencil Problem #7** ✎
7. The percentage of adult height attained by a boy who is x years old can be modeled by $f(x) = 29 + 48.8\log(x+1)$, where x represents the boy's age and $f(x)$ represents the percentage of his adult height. Approximately what percentage of his adult height has a boy attained at age ten?	**7.** The percentage of adult height attained by a girl who is x years old can be modeled by $f(x) = 62 + 35\log(x-4)$, where x represents the girl's age and $f(x)$ represents the percentage of her adult height. Approximately what percentage of her adult height has a girl attained at age 13?

$f(x) = 29 + 48.8\log(x+1)$

$f(10) = 29 + 48.8\log(10+1)$

$\qquad = 29 + 48.8\log 11$

$\qquad \approx 80$

A 10-year-old boy has attained approximately 80% of his adult height.

Objective #8: Use natural logarithms.

✔ **Solved Problem #8**	✎ **Pencil Problem #8** ✎
8a. Find the domain of $f(x) = \ln(4-x)$.	**8a.** Find the domain of $g(x) = \ln(x+2)$.

The domain of f consists of all x for which $4 - x > 0$.

$4 - x > 0$

$\quad -x > -4$

$\quad\quad x < 4$

The domain of f is $(-\infty, 4)$.

8b. The function $f(x) = 13.4\ln x - 11.6$ models the temperature increase, $f(x)$, in an enclosed vehicle after x minutes when the outside air temperature is between 72°F and 96°F. Use the function to find the temperature increase, to the nearest degree, after 30 minutes.

Evaluate the function at 30.

$f(x) = 13.4\ln x - 11.6$

$f(30) = 13.4\ln 30 - 11.6$

≈ 34

The temperature will increase by about 34° after 30 minutes.

8b. The function $f(x) = 5.1\ln x + 19$ models the percentage of U.S. adults putting off medical treatment because of expense, $f(x)$, x years after 2000. Use the function to find the percentage of U.S. adults who put off medical treatment in 2019 because of expense. Round to the nearest percent.
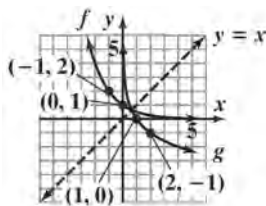
Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $6^y = 216$ *(4.2 #7)*

2. $\log_7 200 = y$ *(4.2 #19)*

3a. -1 *(4.2 #25)*   **3b.** 2 *(4.2 #21)*

4. 0 *(4.2 #37)*

5.

$f(x) = \left(\frac{1}{2}\right)^x$
$g(x) = \log_{1/2} x$
(4.2 #45)

6. $(-\infty, 2)$ *(4.2 #77)*

7. 95.4% *(4.2 #113)*

8a. $(-2, \infty)$ *(4.2 #65)*

8b. about 34% *(4.2 #115a)*

Section 4.3
Properties of Logarithms

How Smart Is This Chimp?

Scientists are often amazed at what a chimpanzee can learn. These scientists are not simply interested in *what* a chimp can learn, but also in a particular chimp's *maximum capacity* for learning, and *how long* the learning takes.

A typical chimpanzee learning sign language can master a maximum of 65 signs. In the Exercise Set of the textbook, one application will take a look at the number of weeks
it will take for a chimp to master 30 signs.

Objective #1: Use the product rule.

✔ *Solved Problem #1*

1. Use the product rule to expand: $\log(100x)$.

$$\log(100x) = \log 100 + \log x$$
$$= \log 10^2 + \log x$$
$$= 2 + \log x$$

✎ *Pencil Problem #1*✎

1. Use the product rule to expand: $\log_5 (7 \cdot 3)$.

Objective #2: Use the quotient rule.

✔ *Solved Problem #2*

2. Use the quotient rule to expand: $\ln\left(\frac{e^5}{11}\right)$.

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11$$
$$= 5 - \ln 11$$

✎ *Pencil Problem #2*✎

2. Use the quotient rule to expand: $\log_4\left(\frac{64}{y}\right)$.

Objective #3: Use the power rule.

✔ *Solved Problem #3*

3a. Use the power rule to expand: $\log_6 3^9$

$$\log_6 3^9 = 9\log_6 3$$

✎ *Pencil Problem #3*✎

3a. Use the power rule to expand: $\log_b x^3$.

3b. Use the power rule to expand: $\log(x+4)^2$

$\log(x+4)^2 = 2\log(x+4)$

3b. Use the power rule to expand: $\log N^{-6}$

Objective #4: Expand logarithmic expressions.

✔ *Solved Problem #4*	✎ *Pencil Problem #4* ✎

4a. Use logarithmic properties to expand: $\ln\sqrt[3]{x}$.

$$\ln\sqrt[3]{x} = \ln x^{\frac{1}{3}}$$
$$= \frac{1}{3}\ln x$$

4a. Use logarithmic properties to expand: $\ln\sqrt[5]{x}$.

4b. Use logarithmic properties to expand:
$\log_b\left(x^4\sqrt[3]{y}\right)$.

$$\log_b\left(x^4\sqrt[3]{y}\right) = \log_b x^4 + \log_b \sqrt[3]{y}$$
$$= \log_b x^4 + \log_b y^{\frac{1}{3}}$$
$$= 4\log_b x + \frac{1}{3}\log_b y$$

4b. Use logarithmic properties to expand: $\log_b(x^2 y)$.

4c. Use logarithmic properties to expand: $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$.

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(\frac{x^{\frac{1}{2}}}{25y^3}\right)$$
$$= \log_5 x^{\frac{1}{2}} - \log_5\left(25y^3\right)$$
$$= \log_5 x^{\frac{1}{2}} - \left(\log_5 25 + \log_5 y^3\right)$$
$$= \log_5 x^{\frac{1}{2}} - \log_5 25 - \log_5 y^3$$
$$= \frac{1}{2}\log_5 x - 2 - 3\log_5 y$$

4c. Use logarithmic properties to expand: $\log_6\left(\frac{36}{\sqrt{x+1}}\right)$.

Objective #5: Condense logarithmic expressions.	
✔ *Solved Problem #5*	✎ *Pencil Problem #5* ✐

5a. Write as a single logarithm: $\log 25 + \log 4$.

$\log 25 + \log 4 = \log(25 \cdot 4)$
$\qquad\qquad = \log 100$
$\qquad\qquad = 2$

5a. Write as a single logarithm: $\log 5 + \log 2$.

5b. Write as a single logarithm: $2\ln x + \dfrac{1}{3}\ln(x+5)$.

$2\ln x + \dfrac{1}{3}\ln(x+5) = \ln x^2 + \ln(x+5)^{\frac{1}{3}}$
$\qquad\qquad\qquad = \ln x^2 + \ln\sqrt[3]{x+5}$
$\qquad\qquad\qquad = \ln\left(x^2 \sqrt[3]{x+5}\right)$

5b. Write as a single logarithm: $4\ln(x+6) - 3\ln x$.

5c. Write as a single logarithm:
$\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$.

$\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$

$= \log_b x^{\frac{1}{4}} - \log_b 5^2 - \log_b y^{10}$

$= \log_b x^{\frac{1}{4}} - \left(\log_b 5^2 + \log_b y^{10}\right)$

$= \log_b \sqrt[4]{x} - \left(\log_b 25 + \log_b y^{10}\right)$

$= \log_b \sqrt[4]{x} - \left(\log_b 25y^{10}\right)$

$= \log_b\left(\dfrac{\sqrt[4]{x}}{25y^{10}}\right)$

5c. Write as a single logarithm:
$3\ln x + 5\ln y - 6\ln z$.

Objective #6: Use the change-of-base property.	

✔ **Solved Problem #6**	✎ **Pencil Problem #6** ✎
6a. Use common logarithms to evaluate $\log_7 2506$.	**6a.** Use common logarithms to evaluate $\log_{0.1} 17$.
$\log_7 2506 = \dfrac{\log 2506}{\log 7}$ ≈ 4.02	
6b. Use natural logarithms to evaluate $\log_7 2506$.	**6b.** Use natural logarithms to evaluate $\log_{0.1} 17$.
$\log_7 2506 = \dfrac{\ln 2506}{\ln 7}$ ≈ 4.02	

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $\log_5 7 + \log_5 3$ *(4.3 #1)*

2. $3 - \log_4 y$ *(4.3 #11)*

3a. $3\log_b x$ *(4.3 #15)*   **3b.** $-6\log N$ *(4.3 #17)*

4a. $\dfrac{1}{5}\ln x$ *(4.3 #19)*   **4b.** $2\log_b x + \log_b y$ *(4.3 #21)*   **4c.** $2 - \dfrac{1}{2}\log_6(x+1)$ *(4.3 #25)*

5a. 1 *(4.3 #41)*   **5b.** $\ln\left[\dfrac{(x+6)^4}{x^3}\right]$ *(4.3 #59)*   **5c.** $\ln\left(\dfrac{x^3 y^5}{z^6}\right)$ *(4.3 #61)*

6a. -1.2304 *(4.3 #75)*   **6b.** -1.2304 *(4.3 #75)*

Under the Sea !

Though it can be pitch black in the depths of the ocean, sunlight is visible as you get closer to the surface. About 12% of the surface sunlight reaches a depth of 20 feet and about 1.6% reaches to a depth of 100 feet.

In the applications of this section, you will use an exponential function to determine the depths that correspond to various percentages of light.

Objective #1: Use like bases to solve exponential equations.

✔ *Solved Problem #1*	✎ *Pencil Problem #1*
1a. Solve: $5^{3x-6} = 125$.	**1a.** Solve: $4^{2x-1} = 64$.

$5^{3x-6} = 125$

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$\quad 3x = 9$

$\quad\; x = 3$

The solution set is $\{3\}$.

1b. Solve: $8^{x+2} = 4^{x-3}$.

1b. Solve: $8^{x+3} = 16^{x-1}$.

$\quad 8^{x+2} = 4^{x-3}$

$(2^3)^{x+2} = (2^2)^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3(x+2) = 2(x-3)$

$\quad 3x + 6 = 2x - 6$

$\quad\; x + 6 = -6$

$\quad\quad\; x = -12$

The solution set is $\{-12\}$.

> ***Objective #2:*** Use logarithms to solve exponential equations.

| ✔ *Solved Problem #2* | ✎ *Pencil Problem #2* ✐ |

2a. Solve: $10^x = 8000$.

Take the common log of both sides of the equation.
$$10^x = 8000$$
$$\log 10^x = \log 8000$$
$$x = \log 8000$$
$$x \approx 3.90$$

The solution set is $\{\log 8000 \approx 3.90\}$.

2a. Solve: $10^x = 3.91$.

2b. Solve: $7e^{2x} - 5 = 58$.

Isolate the exponential expression, then take the natural log of both sides of the equation.

$$7e^{2x} - 5 = 58$$
$$7e^{2x} = 63$$
$$e^{2x} = 9$$
$$\ln e^{2x} = \ln 9$$
$$2x = \ln 9$$
$$x = \frac{\ln 9}{2}$$
$$x = \frac{\ln 3^2}{2}$$
$$x = \frac{2\ln 3}{2}$$
$$x = \ln 3$$
$$x \approx 1.10$$

The solution set is $\{\ln 3 \approx 1.10\}$.

2b. Solve: $7^{x+2} = 410$.

| **Objective #3:** Use exponential form to solve logarithmic equations. |

| ✔ *Solved Problem #3* | 🖋 *Pencil Problem #3* 🖋 |

3a. Solve: $4\ln(3x) = 8$.

$4\ln(3x) = 8$

$\ln(3x) = 2$

$e^2 = 3x$

$\dfrac{e^2}{3} = x$

Check:

$4\ln(3x) = 8$

$4\ln\left[3\left(\dfrac{e^2}{3}\right)\right] = 8$

$4\ln\left(e^2\right) = 8$

$4\cdot 2 = 8$

$8 = 8$, true

The solution set is $\left\{\dfrac{e^2}{3}\right\}$.

3a. Solve: $\log_4(x+5) = 3$.

3b. Solve: $\log x + \log(x-3) = 1$.

$\log x + \log(x-3) = 1$

$\log(x^2 - 3x) = 1$

$10^1 = x^2 - 3x$

$0 = x^2 - 3x - 10$

$0 = (x+2)(x-5)$

$x+2 = 0 \quad$ or $\quad x-5 = 0$

$x = -2 \qquad\qquad x = 5$

Check -2:

$\log x + \log(x-3) = 1$

$\log(-2) + \log(-2-3) = 1$

-2 does not check.

Check 5:

$\log x + \log(x-3) = 1$

$\log 5 + \log(5-3) = 1$

$\log 5 + \log 2 = 1$

$\log 10 = 1$

$1 = 1$, true

The solution set is $\{5\}$.

3b. Solve: $\log_5 x + \log_5(4x-1) = 1$.

Objective #4: Use the one-to-one property of logarithms to solve logarithmic equations.

✔ *Solved Problem #4*

4. Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$

$$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$$

$$x-3 = \frac{7x-23}{x+1}$$

$$(x+1)(x-3) = (x+1)\frac{7x-23}{x+1}$$

$$x^2 - 2x - 3 = 7x - 23$$

$$x^2 - 9x + 20 = 0$$

$$(x-4)(x-5) = 0$$

$$x-4 = 0 \quad \text{or} \quad x-5 = 0$$

$$x = 4 \qquad\qquad x = 5$$

Check 4:
$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$
$$\ln(4-3) = \ln(7 \cdot 4 - 23) - \ln(4+1)$$
$$\ln 1 = \ln 5 - \ln 5$$
$$0 = 0, \text{ true}$$

Check 5:
$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$
$$\ln(5-3) = \ln(7 \cdot 5 - 23) - \ln(5+1)$$
$$\ln 2 = \ln 12 - \ln 6$$
$$\ln 2 = \ln\left(\frac{12}{6}\right)$$
$$\ln 2 = \ln 2, \text{ true}$$

The solution set is $\{4,5\}$.

✎ *Pencil Problem #4*

4. Solve: $\log(x+4) - \log 2 = \log(5x+1)$

Objective #5: Solve applied problems involving exponential and logarithmic equations.	

✔ *Solved Problem #5*	✏ *Pencil Problem #5*
5a. Medical research indicates that the risk of having a car accident increases exponentially as the concentration of alcohol in the blood increases. The risk is modeled by $R = 6e^{12.77x}$ where x is the blood alcohol concentration and R, given as a percent, is the risk of having a car accident. What blood alcohol concentration corresponds to a 7% risk of a car accident?	**5a.** The formula $A = 37.3e^{0.0095e}$ models the population of California, A, in millions, t years after 2010. When did the population of California reach 40 million?

$$R = 6e^{12.77x}$$
$$6e^{12.77x} = 7$$
$$e^{12.77x} = \frac{7}{6}$$
$$\ln e^{12.77x} = \ln \frac{7}{6}$$
$$12.77x = \ln \frac{7}{6}$$
$$x = \frac{\ln \frac{7}{6}}{12.77}$$
$$x \approx 0.01$$

For a blood alcohol concentration of 0.01, the risk of a car accident is 7%.

5b. How long, to the nearest tenth of a year, will it take $1000 to grow to $3600 at an annual rate of 8% compounded quarterly?	**5b.** How long, to the nearest tenth of a year, will it take $35,000 to grow to $70,000 at 8% annual interest compounded continuously?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$3600 = 1000\left(1 + \frac{0.08}{4}\right)^{4t}$$
$$3.6 = 1.02^{4t}$$
$$1.02^{4t} = 3.6$$
$$\ln 1.02^{4t} = \ln 3.6$$
$$4t \ln 1.02 = \ln 3.6$$
$$t = \frac{\ln 3.6}{4 \ln 1.02}$$
$$t \approx 16.2$$

After approximately 16.2 years, the $1000 will grow to $3600.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $\{2\}$ *(4.4 #7)*   **1b.** $\{13\}$ *(4.4 #19)*

2a. $\{\log 3.91 \approx 0.59\}$ *(4.4 #23)*

2b. $\left\{ \dfrac{\ln 410}{\ln 7} - 2 \approx 1.09 \right\}$ *(4.4 #37)*

3a. $\{59\}$ *(4.4 #53)*

3b. $\left\{ \dfrac{5}{4} \right\}$; note: reject -1 *(4.4 #67)*

4. $\left\{ \dfrac{2}{9} \right\}$ *(4.4 #83)*

5a. 2017   *(4.4 #103b)*

5b. 8.7 years   *(4.4 #111)*

NOT Tooth Decay!

One of algebra's many applications is to predict the behavior of variables.
This can be done with *exponential growth* and *decay models*. With
exponential growth or decay, quantities grow or decay at a rate directly
proportional to their size. Populations that are growing exponentially grow
extremely rapidly as they get larger because there are more adults to have
offspring.

In this section we explore how to create such functions
and how to use them to make predictions.

Objective #1: Model exponential growth and decay.

✔ Solved Problem #1

1. In 2000, the population of Africa was 807 million and by 2020 it had grown to 1341 million.

1a. Use the exponential growth model $A = A_0 e^{kt}$, in which t is the number of years after 2000, to find the exponential growth function that models the data.

2020 is 20 years after 2000.
Thus, when $t = 20$, $A = 1341$.
$A_0 = 807$

Substitute these values to find k.

$A = A_0 e^{kt}$

$1341 = 807 e^{k(20)}$

Solve for k.

$$1341 = 807 e^{k(20)}$$
$$\frac{1341}{807} = e^{20k}$$
$$\ln\left(\frac{1341}{807}\right) = \ln e^{20k}$$
$$\ln\left(\frac{1341}{807}\right) = 20k$$
$$k = \frac{\ln\left(\frac{1341}{807}\right)}{20} \approx 0.025$$

Thus, the growth function is $A = 807 e^{0.025t}$.

✎ Pencil Problem #1 ✎

1. In 2020, the population of Israel was approximately 8.7 million and by 2050 it is projected to grow to 12.7 million.

1a. Use the exponential growth model $A = A_0 e^{kt}$, in which t is the number of years after 2020, to find an exponential growth function that models the data.

1b. By which year will Africa's population reach 2000 million, or two billion?

$$A = 807e^{0.025t}$$
$$2000 = 807e^{0.025t}$$
$$\frac{2000}{807} = e^{0.025t}$$
$$\ln\left(\frac{2000}{807}\right) = \ln e^{0.025t}$$
$$\ln\left(\frac{2000}{807}\right) = 0.025t$$
$$t = \frac{\ln\left(\dfrac{2000}{807}\right)}{0.025} \approx 36$$

Africa's population will reach 2000 million approximately 36 years after 2000, or 2036.

1b. In which year will Israel's population be 10 million?

Objective #2: Use logistic growth models.

✔ *Solved Problem #2*

2. In a learning theory project, psychologists discovered that

$$f(t) = \frac{0.8}{1 + e^{-0.2t}}$$

is a model for describing the proportion of correct responses, $f(t)$, after t learning trials.

2a. Find the proportion of correct responses after 10 learning trials.

We substitute 10 for t in the logistic growth function.
$$f(10) = \frac{0.8}{1 + e^{-0.2(10)}} \approx 0.7$$
The proportion of correct responses after 10 trials is approximately 0.7.

✎ *Pencil Problem #2*

2. The logistic growth function

$$f(t) = \frac{100,000}{1 + 5000e^{-t}}$$

describes the number of people, $f(t)$, who have become ill with influenza t weeks after its initial outbreak in a particular community.

2a. How many people were ill by the end of the fourth week?

Copyright © 2022 Pearson Education Inc.

2b. What is the limiting size of $f(t)$, the proportion of correct responses, as continued learning trials take place?

The number in the numerator of the logistic growth function, 0.8, is the limiting size of the proportion of correct responses.

**2b.** What is the limiting size of the population that becomes ill?

Objective #3: Choose an appropriate model for data.

✔ Solved Problem #3

3. The table shows the populations of various cities and the average walking speed of a person living in the city. Create a scatter plot for the data. Based on the scatter plot, what type of function would be a good choice for modeling the data?

Population (thousands)	Walking Speed (feet per second)
5.5	0.6
14	1.0
71	1.6
138	1.9
342	2.2

Scatter plot:
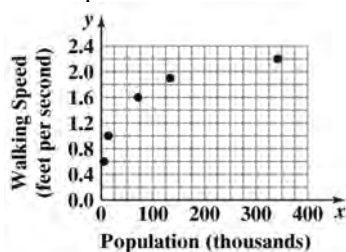
A logarithmic function would be a good choice for modeling the data.

✎ Pencil Problem #3

3. The table shows the percentage of miscarriages by women of various ages. Create a scatter plot for the data. Determine the type of function that would be a good choice for modeling the data?

Woman's Age	Percent of Miscarriages
22	9%
27	10%
32	13%
37	20%
42	38%
47	52%

Objective #4: Express an exponential model in base e.

✔ **Solved Problem #4**	🖉 **Pencil Problem #4** 🖉
4. Rewrite $y = 4(7.8)^x$ in terms of base e. Express the answer in terms of a natural logarithm and then round to three decimal places.	**4.** Rewrite $y = 100(4.6)^x$ in terms of base e. Express the answer in terms of a natural logarithm and then round to three decimal places.

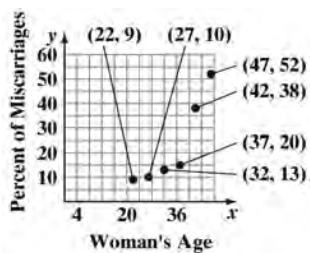$y = 4(7.8)^x$

$\quad = 4e^{(\ln 7.8)x}$

Rounded to three decimal places:

$y = 4e^{2.054x}$

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $A = 8.7e^{0.013t}$ *(4.5 #7a)*

1b. 2031   *(4.5 #7b)*

2a. approximately 1080 people   *(4.5 #37b)*

2b. 100,000 people   *(4.5 #37c)*

3.   exponential function   *(12.5 #47)*

4. $y = 100e^{(\ln 4.6)x} = 100e^{1.526x}$ *(4.5 #53)*