Section 3.1
Quadratic Functions and Their Graphs

Heads UP!!!

Many sports involve objects that are thrown, kicked, or hit, and then proceed with no additional force of their own.
Such objects are called projectiles.

In this section of your textbook, you will learn to use graphs of quadratic functions to gain a visual understanding of various projectile sports.

Objective #1: Recognize characteristics of parabolas.

✔ *Solved Problem #1*

1. True or false: The *vertex* of a parabola is also called the *turning point*.

True

Pencil Problem #1

1. True or false: The *vertex* of a parabola is always the minimum point of the parabola.

Objective #2: Graph parabolas.

✔ *Solved Problem #2*

2a. Graph the quadratic function: $f(x) = -(x-1)^2 + 4$

Since $a = -1$ is negative, the parabola opens downward.
The vertex of the parabola is $(h, k) = (1, 4)$.

Replace $f(x)$ with 0 to find x–intercepts.

$$0 = -(x-1)^2 + 4$$
$$(x-1)^2 = 4$$
$$x - 1 = \pm\sqrt{4}$$
$$x - 1 = \pm 2$$
$$x - 1 = 2 \quad \text{or} \quad x - 1 = -2$$
$$x = 3 \qquad\qquad x = -1$$

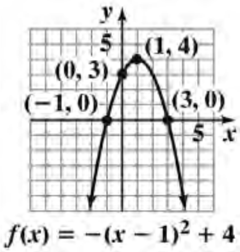
The x–intercepts are -1 and 3.

Pencil Problem #2

2a. Graph the quadratic function: $f(x) = (x-4)^2 - 1$

Set $x = 0$ and solve for *y* to obtain the *y*–intercept.

$y = -(0-1)^2 + 4 = 3$



2b. Graph the quadratic function $f(x) = -x^2 + 4x + 1$. Use the graph to identify the function's domain and its range.

Since $a = -1$ is negative, the parabola opens downward.

The *x*–coordinate of the vertex of the parabola is

$-\frac{b}{2a} = -\frac{4}{2(-1)} = -\frac{4}{-2} = 2.$

The *y*–coordinate of the vertex of the parabola is

$f\left(-\frac{b}{2a}\right) = f(2) = -(2)^2 + 4(2) + 1 = 5.$

The vertex is $(2,5)$.

Replace $f(x)$ with 0 to find *x*–intercepts.

$0 = -x^2 + 4x + 1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(1)}}{2(-1)}$

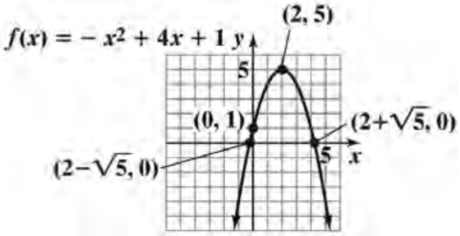
$x = 2 \pm \sqrt{5}$

$x \approx -0.2$ or $x \approx 4.2$

The *x*–intercepts are -0.2 and 4.2.

Set $x = 0$ and solve for *y* to obtain the *y*–intercept.

$y = -0^2 + 4 \cdot 0 + 1 = 1$



Domain: $(-\infty, \infty)$ Range: $(-\infty, 5]$

2b. Graph the quadratic function $f(x) = x^2 + 3x - 10$. Use the graph to identify the function's range.

Objective #3: Determine a quadratic function's minimum or maximum value.

✔ Solved Problem #3

3. Consider the quadratic function $f(x)=4x^2-16x+1000.$

3a. Determine, without graphing, whether the function has a minimum value or a maximum value.

Because $a>0,$ the function has a minimum value.

3b. Find the minimum or maximum value and determine where it occurs.

The minimum value occurs at $-\frac{b}{2a}=-\frac{-16}{2(4)}=2.$

The minimum of $f(x)$ is $f(2)=4\cdot 2^2-16\cdot 2+1000$
$=984.$

3c. Identify the function's domain and its range.

Like all quadratic functions, the domain is $(-\infty,\infty).$

Because the minimum is 984, the range includes all real numbers at or above 984. The range is $\left[984,\infty\right).$

Pencil Problem #3

3. Consider the quadratic function $f(x)=-4x^2+8x-3.$

3a. Determine, without graphing, whether the function has a minimum value or a maximum value.

3b. Find the minimum or maximum value and determine where it occurs.

3c. Identify the function's domain and its range.

Objective #4: Solve problems involving a quadratic function's minimum or maximum value.

✔ Solved Problem #4

4. Among all pairs of numbers whose difference is 8, find a pair whose product is as small as possible. What is the minimum product?

Let the two numbers be represented by *x* and *y*, and let the product be represented by *P*.

We must minimize $P = xy$.

Because the difference of the two numbers is 8, then $x-y=8.$

Solve for *y* in terms of *x*.

$$x-y=8$$
$$-y=-x+8$$
$$y=x-8$$

Write *P* as a function of *x*.

Pencil Problem #4

4. Among all pairs of numbers whose sum is 16, find a pair whose product is as large as possible. What is the maximum product?

$P = xy$

$P(x) = x(x-8)$

$P(x) = x^2 - 8x$

Because $a > 0$, the function has a minimum value that occurs at $x = -\dfrac{b}{2a}$

$= -\dfrac{-8}{2(1)}$

$= 4.$

Substitute to find the other number.

$y = x - 8$

$y = 4 - 8$

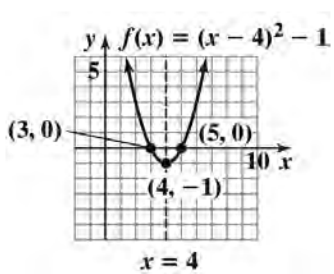
$= -4$

The two numbers are 4 and -4.

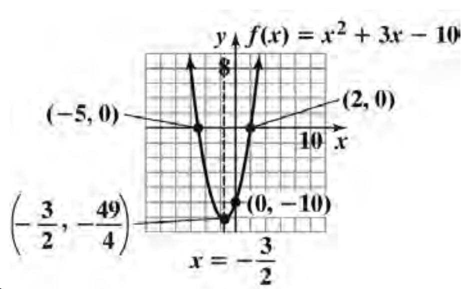
The minimum product is $P = xy = (4)(-4) = -16$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. false *(3.1 #41)*

2a. 

(3.1 #17)

2b. 

Range: $\left[-\dfrac{49}{4}, \infty\right)$ *(3.1 #29)*

3a. maximum *(3.1 #41a)*

3b. The maximum is 1 at $x = 1$. *(3.1 #41b)*

3c. Domain: $(-\infty, \infty)$; Range: $(-\infty, 1]$ *(3.1 #41c)*

4. The maximum product is 64 when the numbers are 8 and 8. *(3.1 #61)*

Section 3.2
Polynomial Functions and Their Graphs

Pay at the Pump !

Other than outrage, what is going on at the gas pumps?
Is surging demand creating the increasing oil prices?
Like all things in a free market economy, the price of a commodity is based on supply and demand.

In the Exercise Set for this section, we will explore the volatility of gas prices over the past several years.

Objective #1: Identify polynomial functions.

✔ *Solved Problem #1*

1. The exponents on the variables in a polynomial function must be nonnegative integers.

True

Pencil Problem #1

1. The coefficients of the variables in a polynomial function must be nonnegative integers.

Objective #2: Recognize characteristics of graphs of polynomial functions.

✔ *Solved Problem #2*

2. The graph of a polynomial function may have a sharp corner.

False. The graphs of polynomial functions are smooth, meaning that they have rounded curves and no sharp corners.

Pencil Problem #2

2. The graph of a polynomial function may have a gap or break.

Objective #3: Determine end behavior.

✔ *Solved Problem #3*

3. Use the Leading Coefficient Test to determine the end behavior of the graph of each function.

3a. $f(x) = x^4 - 4x^2$

The term with the greater exponent is x^4, or $1x^4$. The leading coefficient is 1, which is positive. The degree of the function is 4, which is even. Even-degree polynomial functions have the same behavior at each end. Since the leading coefficient is positive, the graph rises to the left and rises to the right.

Pencil Problem #3

3. Use the Leading Coefficient Test to determine the end behavior of the graph of each function.

3a. $f(x) = 5x^3 + 7x^2 - x + 9$

3b. $f(x) = 2x^3(x-1)(x+5)$

The function is in factored form, but we can determine the degree and the leading coefficient without multiplying it out. The factors $2x^3$, $x - 1$, and $x + 5$ are of degree 3, 1, and 1, respectively. When we multiply expressions with the same base, we add exponents, so the degree of the function is 3 + 1 + 1, or 5, which is odd. Without multiplying out, you should be able to see that the leading coefficient is 2, which is positive.

Odd-degree polynomial functions have graphs with opposite behavior at each end. Since the leading coefficient is positive, the graph falls to the left and rises to the right.

3b. $f(x) = -x^2(x-1)(x+3)$

Objective #4: Use factoring to find zeros of polynomial functions.

✔ *Solved Problem #4*

4. Find all zeros of $f(x) = x^3 + 2x^2 - 4x - 8$.

Set $f(x)$ equal to zero.

$$x^3 + 2x^2 - 4x - 8 = 0$$
$$x^2(x+2) - 4(x+2) = 0$$
$$(x+2)(x^2 - 4) = 0$$
$$(x+2)(x+2)(x-2) = 0$$

Apply the zero-product principle.

$x + 2 = 0$ or $x + 2 = 0$ or $x - 2 = 0$

$x = -2$ $\quad$ $x = -2$ $\quad$ $x = 2$

The zeros are -2 and 2.

Pencil Problem #4

4. Find all zeros of $f(x) = x^3 + 2x^2 - x - 2$.

Objective #5: Identify zeros and their multiplicities.

✔ *Solved Problem #5*

5. Find the zeros of $f(x) = -4\left(x + \frac{1}{2}\right)^2 (x-5)^3$ and give the multiplicity of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero.

Set each factor equal to zero.

$x + \frac{1}{2} = 0$ or $x - 5 = 0$

$x = -\frac{1}{2}$ $\quad$ $x = 5$

$-\frac{1}{2}$ is a zero of multiplicity 2, and 5 is a zero of multiplicity 3.

Because the multiplicity of $-\frac{1}{2}$ is even, the graph touches the x-axis and turns around at this zero.

Because the multiplicity of 5 is odd, the graph crosses the x-axis at this zero.

Pencil Problem #5

5. Find the zeros of $f(x) = 4(x-3)(x+6)^3$ and give the multiplicity of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero.

Objective #6: Use the Intermediate Value Theorem.

✔ *Solved Problem #6*

6. Show that the polynomial function $f(x)=3x^3-10x+9$ has a real zero between -3 and -2.

Evaluate f at -3 and -2.

$f(-3)=3(-3)^3-10(-3)+9=-42$

$f(-2)=3(-2)^3-10(-2)+9=5$

The sign change between $f(-3)$ and $f(-2)$ shows that f has a real zero between -3 and -2.

Pencil Problem #6

6. Show that the polynomial function $f(x)=2x^4-4x^2+1$ has a real zero between -1 and 0.

Objective #7: Understand the relationship between degree and turning points.

✔ *Solved Problem #7*

7. If a polynomial function, f, is of degree 5, what is the greatest number of turning points on its graph?

The greatest number of turning points on the graph of a polynomial of degree 5 is $5-1$, or 4.

Pencil Problem #7

7. If a polynomial function, f, is of degree 4, what is the greatest number of turning points on its graph?

Objective #8: Graph polynomial functions.

✔ *Solved Problem #5*

8. Use the five-step strategy to graph $f(x)=x^3-3x^2$.

Step 1: Determine end behavior.
Since $f(x)=x^3-3x^2$ is an odd-degree polynomial and since the leading coefficient, 1, is positive, the graph falls to the left and rises to the right.

Step 2: Find x-intercepts by setting $f(x)=0$.

$x^3-3x^2=0$

$x^2(x-3)=0$

Apply the zero-product principle.

$x^2=0$ or $x-3=0$

$x=0$ $\quad$ $x=3$

The zeros of f are 0 and 3. The graph touches the x-axis at 0 since it has multiplicity 2. The graph crosses the x-axis at 3 since it has multiplicity 1.

Pencil Problem #5

8. Use the five-step strategy to graph $f(x)=x^4-9x^2$.

Step 3: Find the *y*-intercept by computing $f(0)$.

$$f(x) = x^3 - 3x^2$$
$$f(0) = 0^3 - 3(0)^2$$
$$= 0$$

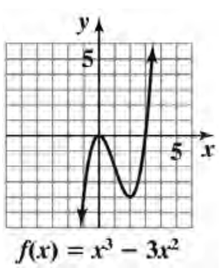
There is a *y*-intercept at 0, so the graph passes through (0, 0).

Step 4: Use possible symmetry to help draw the graph.

$$f(x) = x^3 - 3x^2$$
$$f(-x) = (-x)^3 - 3(-x)^2$$
$$= -x^3 - 3x^2$$

Since $f(-x) \neq f(x)$ and since $f(-x) \neq -f(x)$, the function is neither even nor odd, and the graph is neither symmetric with respect to the *y*-axis nor the origin.

Step 5: Draw the graph.



Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. False *(3.2 #3)* **2.** False *(3.2 #13)*

3a. The graph falls to the left and rises to the right. *(3.2 #19)*

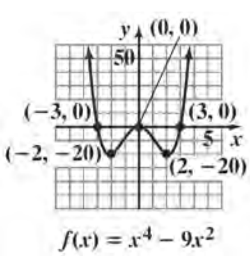
3b. The graph falls to the left and falls to the right. *(3.2 #59a)*

4. -2, -1, and 1 *(3.2 #41b)*

5. zeros: 3 (multiplicity 1) and -6 (multiplicity 3); The graph crosses the *x*-axis at 3 and at -6. *(3.2 #27)*

6. $f(-1) = -1$ and $f(0) = 1$; The sign change between $f(-1)$ and $f(0)$ shows that f has a real zero between -1 and 0. *(3.2 #35)*

7. 3 *(3.2 #47e)*

8. 

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Section 3.3
Dividing Polynomials; Remainder and Factor Theorems

What Happened to My Sweater?

It's that first brisk morning in autumn and you go to the closet for your favorite sweater. But what's that? There's a hole. No. There are dozens of holes.
In this section's Exercise Set, you will work with a polynomial function that models the number of eggs in a female moth based on her abdominal width. The techniques of this section provide a new way of evaluating the function to find out how many moths were eating your sweater.

Objective #1: Use long division to divide polynomials.

✔ Solved Problem #1

1. Divide $2x^4+3x^3-7x-10$ by x^2-2x.

Rewrite the dividend with the missing power of *x* and divide.

$$
\begin{array}{r}
2x^2+7x+14 \\
x^2-2x\overline{\big)\,2x^4+3x^3+0x^2-7x-10} \\
\underline{2x^4-4x^3} \qquad\qquad\qquad \\
7x^3+0x^2 \qquad\qquad \\
\underline{7x^3-14x^2} \qquad\qquad \\
14x^2-7x \qquad \\
\underline{14x^2-28x} \qquad \\
21x-10
\end{array}
$$

Thus, $\dfrac{2x^4+3x^3-7x-10}{x^2-2x}=2x^2+7x+14+\dfrac{21x-10}{x^2-2x}$

Pencil Problem #1

1. Divide $4x^4-4x^2+6x$ by $x-4$ using long division.

Objective #2: Use synthetic division to divide polynomials.

✔ Solved Problem #2

2. Use synthetic division: $(x^3-7x-6)\div(x+2)$

$$
\begin{array}{r|rrrr}
-2 & 1 & 0 & -7 & -6 \\
 & & -2 & 4 & 6 \\
\hline
 & 1 & -2 & -3 & 0
\end{array}
$$

Thus, $(x^3-7x-6)\div(x+2)=x^2-2x-3$

Pencil Problem #2

2. Use synthetic division: $(3x^2+7x-20)\div(x+5)$

Objective #3: Evaluate a polynomial function using the Remainder Theorem.

✔ Solved Problem #3

3. Given $f(x)=3x^3+4x^2-5x+3$, use the Remainder Theorem to find $f(-4)$.

$$\begin{array}{r|rrrr} -4 & 3 & 4 & -5 & 3 \\ & & -12 & 32 & -108 \\ \hline & 3 & -8 & 27 & -105 \end{array} \leftarrow f(-4)=-105$$

Pencil Problem #3

3. Given $f(x)=2x^3-11x^2+7x-5$, use the Remainder Theorem to find $f(4)$.

Objective #4: Use the Factor Theorem to solve a polynomial equation.

✔ Solved Problem #4

4. Solve the equation $15x^3+14x^2-3x-2=0$ given that -1 is a zero of $f(x)=15x^2+14x^2-3x-2$.

Synthetic division verifies that $x+1$ is a factor.

$$\begin{array}{r|rrrr} -1 & 15 & 14 & -3 & -2 \\ & & -15 & 1 & 2 \\ \hline & 15 & -1 & -2 & 0 \end{array}$$

Next, continue factoring to find all solutions.

$$15x^3+14x^2-3x-2=0$$
$$(x+1)(15x^2-x-2)=0$$
$$(x+1)(5x-2)(3x+1)=0$$
$$x+1=0 \quad \text{or} \quad 5x-2=0 \quad \text{or} \quad 3x+1=0$$
$$x=-1 \qquad 5x=2 \qquad 3x=-1$$
$$x=\frac{2}{5} \qquad x=-\frac{1}{3}$$

The solution set is $\left\{-1,-\frac{1}{3},\frac{2}{5}\right\}$

Pencil Problem #4

4. Solve the equation $2x^3-5x^2+x+2=0$ given that 2 is a zero of $f(x)=2x^3-5x^2+x+2$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $4x^3+16x^2+60x+246+\frac{984}{x-4}$ *(3.3 #11)*

2. $3x-8+\frac{20}{x+5}$ *(3.3 #19)*

3. -25 *(3.3 #33)*

4. $\left\{-\frac{1}{2},\ 1,\ 2\right\}$ *(3.3 #43)*

Section 3.4
Zeros of Polynomials

Do I Have to Check My Bag?

Airlines have regulations on the sizes of carry-on luggage that are allowed. As a passenger, you are interested in the volume of your luggage, but the airline is concerned about the sum of bag's length, width, and depth.
In this section's Exercise Set, you will work with a polynomial function that relates the two quantities and allows you to find dimensions of a carry-on bag that meet both your volume requirement and the airline's regulations.

Objective #1: Use the Rational Zero Theorem to find possible rational zeros.

✔ *Solved Problem #1*

1. List all possible rational zeros of $f(x) = 4x^5 + 12x^4 - x - 3$.

Factors of the constant term -3: $\pm 1, \ \pm 3$
Factors of the leading coefficient 4: $\pm 1, \ \pm 2, \ \pm 4$

The possible rational zeros are:

$$\frac{\text{Factors of } -3}{\text{Factors of } 4} = \frac{\pm 1, \ \pm 3}{\pm 1, \ \pm 2, \ \pm 4}$$

$$= \pm 1, \ \pm 3, \ \pm\frac{1}{2}, \ \pm\frac{3}{2}, \ \pm\frac{1}{4}, \ \pm\frac{3}{4}$$

Pencil Problem #1

1. List all possible rational zeros of $f(x) = 3x^4 - 11x^3 - x^2 + 19x + 6$.

Objective #2: Find zeros of a polynomial function.

✔ *Solved Problem #2*

2. Find all zeros of $f(x) = x^3 + x^2 - 5x - 2$.

First, list the possible rational zeros:

$$\frac{\text{Factors of } -2}{\text{Factors of } 1} = \frac{\pm 1, \ \pm 2}{\pm 1} = \pm 1, \ \pm 2$$

Now use synthetic division to find a rational zero from among the list of possible rational zeros. Try 2:

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -5 & -2 \\ & & 2 & 6 & 2 \\ \hline & 1 & 3 & 1 & 0 \end{array}$$

The last number in the bottom row is 0.
Thus 2 is a zero and $x - 2$ is a factor.

The first three numbers in the bottom row of the synthetic division give the coefficients of the other factor. This factor is $x^2 + 3x + 1$.

Pencil Problem #2

2. Find all zeros of $f(x) = x^3 + 4x^2 - 3x - 6$.

Factor completely: $x^3 + x^2 - 5x - 2 = 0$

$(x-2)(x^2+3x+1) = 0$

Since x^2+3x+1 is not factorable, use the quadratic formula to find the remaining zeros.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{-3 \pm \sqrt{5}}{2}$$

The zeros are 2 and $\frac{-3 \pm \sqrt{5}}{2}$.

Objective #3: Solve polynomial equations.

Solved Problem #3

3. Solve: $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$

First, list the possible rational roots:

$$\frac{\text{Factors of } 13}{\text{Factors of } 1} = \frac{\pm 1,\ \pm 13}{\pm 1} = \pm 1,\ \pm 13$$

Now use synthetic division to find a rational root from among the list of possible rational roots. Try 1.

$$\begin{array}{r|rrrrr} 1 & 1 & -6 & 22 & -30 & 13 \\ & & 1 & -5 & 17 & -13 \\ \hline & 1 & -5 & 17 & -13 & 0 \end{array}$$

The last number in the bottom row is 0.
Thus, 1 is a root.

Rewrite the equation in factored form using the bottom row of the synthetic division to obtain the coefficients of the other factor.

$x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$

$(x-1)(x^3 - 5x^2 + 17x - 13) = 0$

Use the same approach to find another root. Try 1 again.

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & 0 \end{array}$$

The last number in the bottom row is 0.
Thus, 1 is a root (of multiplicity 2).

The first three numbers in the bottom row of the synthetic division give the coefficients of the factor $x^2 - 4x + 13$.

Pencil Problem #3

3. Solve: $x^3 - 2x^2 - 11x + 12 = 0$

Since $x^2 - 4x + 13$ is not factorable, use the quadratic formula to find the remaining roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{-36}}{2}$$

$$x = \frac{4 \pm 6i}{2}$$

$$x = 2 \pm 3i$$

The roots are 1 and $2 \pm 3i$.

Objective #4: Use the Linear Factorization Theorem to find polynomials with given zeros.

✔ *Solved Problem #4*

4. Find a third-degree polynomial function $f(x)$ with real coefficients that has -3 and i as zeros such that $f(1) = 8$.

Because i is a zero and the polynomial has real coefficients, the conjugate, $-i$, must also be a zero. We can now use the Linear Factorization Theorem.

$$\begin{aligned} f(x) &= a_n(x - c_1)(x - c_2)(x - c_3) \\ &= a_n(x - (-3))(x - i)(x - (-i)) \\ &= a_n(x + 3)(x - i)(x + i) \\ &= a_n(x + 3)(x^2 - i^2) \\ &= a_n(x + 3)(x^2 - (-1)) \\ &= a_n(x + 3)(x^2 + 1) \\ &= a_n(x^3 + 3x^2 + x + 3) \end{aligned}$$

Now we use $f(1) = 8$ to find a_n.

$$\begin{aligned} f(1) = a_n(1^3 + 3\cdot 1^2 + 1 + 3) &= 8 \\ 8a_n &= 8 \\ a_n &= 1 \end{aligned}$$

Now substitute 1 for a_n in the formula for $f(x)$.

$$f(x) = 1(x^3 + 3x^2 + x + 3)$$

or $f(x) = x^3 + 3x^2 + x + 3$

Pencil Problem #4

4. Find a fourth-degree polynomial function $f(x)$ with real coefficients that has i and $3i$ as zeros such that $f(-1) = 20$.

Objective #5: Use Descartes' Rule of Signs.

✔ Solved Problem #5

5. Determine the possible numbers of positive and negative real zeros of $f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120.$

Count the number of sign changes in $f(x)$.

$f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120$

Since $f(x)$ has four sign changes, it has 4, 2, or 0 positive real zeros.

Count the number of sign changes in $f(-x)$.

$$f(-x) = (-x)^4 - 14(-x)^3 + 71(-x)^2 - 154(-x) + 120$$
$$= x^4 + 14x^3 + 71x^2 + 154x + 120$$

Since $f(-x)$ has no sign changes, $f(x)$ has 0 negative real zeros.

Pencil Problem #5

5. Determine the possible numbers of positive and negative real zeros of $f(x) = x^3 + 2x^2 + 5x + 4.$

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm\frac{1}{3},\ \pm\frac{2}{3}$ *(3.4 #3)*

2. $-1,\ \frac{-3-\sqrt{33}}{3}$, and $\frac{-3+\sqrt{33}}{3}$ *(3.4 #13)*

3. $\{-3,\ 1,\ 4\}$ *(3.4 #17)*

4. $f(x) = x^4 + 10x^2 + 9$ *(3.4 #29)*

5. f has no positive real zeros and either 3 or 1 negative real zeros *(3.4 #33)*

Section 3.5
Rational Functions and Their Graphs

Decreasing Costs with Increased Production?

In a simple business model, the cost, $C(x)$, to produce x units of a product is the sum of the fixed and variable costs and can be expressed in a form similar to $C(x) = \$500{,}000 + \$400x$. In this model, the cost increases by \$400 for each additional unit.
If we divide the cost, $C(x)$, by x, the number of units produced, we obtain the function $\bar{C}(x)$, which represents the average cost of each item. By studying the rational function $\bar{C}(x)$, we'll see that the average cost per item decreases for each additional unit.

Objective #1: Find the domains of rational functions.

✔ Solved Problem #1

1a. Find the domain of $g(x) = \dfrac{x}{x^2 - 25}$.

The denominator of $g(x) = \dfrac{x}{x^2 - 25}$ is 0 when $x = -5$ or $x = 5$. The domain of g consists of all real numbers except -5 and 5. This can be expressed as
$\{x \mid x \neq -5, x \neq 5\}$ or $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$

1b. Find the domain of $h(x) = \dfrac{x+5}{x^2+25}$.

No real numbers cause the denominator of $h(x) = \dfrac{x+5}{x^2+25}$ to equal 0. The domain of h consists of all real numbers, or $(-\infty, \infty)$.

Pencil Problem #1

1a. Find the domain of $h(x) = \dfrac{x+7}{x^2 - 49}$.

1b. Find the domain of $f(x) = \dfrac{x+7}{x^2+49}$.

Objective #2: Use arrow notation.

✔ Solved Problem #2

2. True or false: The notation "$x \to a^+$" means that the values of x are increasing without bound.

False. "$x \to a^+$" means that x is approaching a from the right.

Pencil Problem #2

2. True or false: If $f(x) \to 0$ as $x \to \infty$, then the graph of f approaches the x-axis to the right.

Objective #3: Solve polynomial equations.

✔ Solved Problem #3

3a. Find the vertical asymptotes, if any, of the graph of the rational function: $g(x)=\dfrac{x-1}{x^2-1}$.

The numerator and denominator have a factor in common. Therefore, simplify *g*.

$$g(x)=\frac{x-1}{x^2-1}=\frac{x-1}{(x+1)(x-1)}=\frac{1}{x+1}$$

The only zero of the denominator of the simplified function is -1.

Thus, the line $x=-1$ is a vertical asymptote for the graph of *g*.

3b. Find the vertical asymptotes, if any, of the graph of the rational function: $h(x)=\dfrac{x-1}{x^2+1}$.

The denominator cannot be factored.
The denominator has no real zeros.

Thus, the graph of *h* has no vertical asymptotes.

Pencil Problem #3

3a. Find the vertical asymptotes, if any, of the graph of the rational function: $h(x)=\dfrac{x}{x(x+4)}$.

3b. Find the vertical asymptotes, if any, of the graph of the rational function: $r(x)=\dfrac{x}{x^2+4}$.

Objective #4: Identify horizontal asymptotes.

✔ Solved Problem #4

4a. Find the horizontal asymptotes, if any, of the graph of the rational function: $f(x)=\dfrac{9x^2}{3x^2+1}$.

The degree of the numerator, 2, is equal to the degree of the denominator, 2.
The leading coefficients of the numerator and denominator are 9 and 3, respectively.

Thus, the equation of the horizontal asymptote is $y=\dfrac{9}{3}$ or $y=3$.

Pencil Problem #4

4a. Find the horizontal asymptotes, if any, of the graph of the rational function: $f(x)=\dfrac{-2x+1}{3x+5}$.

4b. Find the horizontal asymptotes, if any, of the graph of the rational function: $h(x)=\dfrac{9x^3}{3x^2+1}$.

The degree of the numerator, 3, is greater than the degree of the denominator, 2.

Thus, the graph of h has no horizontal asymptote.

4b. Find the horizontal asymptotes, if any, of the graph of the rational function: $f(x)=\dfrac{12x}{3x^2+1}$.

Objective #5: Use transformations to graph rational functions.

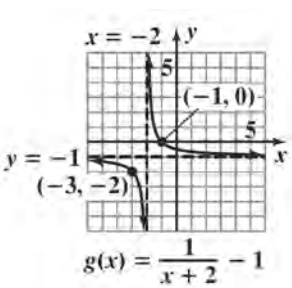
✔ *Solved Problem #5*

5. Use the graph of $f(x)=\dfrac{1}{x}$ to graph $g(x)=\dfrac{1}{x+2}-1$.

Start with the graph of $f(x)=\dfrac{1}{x}$ and two points on its graph, such as (–1, –1) and (1, 1).

First move the graph two units to the left to graph $y=\dfrac{1}{x+2}$; the indicated points end up at (–3, –1) and (–1, 1). The vertical asymptote is now $x = -2$.

Next move the graph down one unit to graph $g(x)=\dfrac{1}{x+2}-1$; the indicated points end up at (–3, –2) and (–1, 0). The horizontal asymptote is now $y = -1$.



Pencil Problem #5

5. Use the graph of $f(x)=\dfrac{1}{x}$ to graph $g(x)=\dfrac{1}{x+1}-2$.

Objective #6: Graph rational functions.

✔ Solved Problem #6

6a. Graph: $f(x)=\dfrac{3x-3}{x-2}$

Step 1: $f(-x)=\dfrac{3(-x)-3}{-x-2}=\dfrac{-3x-3}{-x-2}=\dfrac{3x+3}{x+2}$

Because $f(-x)$ does not equal $f(x)$ or $-f(x)$, the graph has neither *y*-axis symmetry nor origin symmetry.

Step 2: $f(0)=\dfrac{3(0)-3}{0-2}=\dfrac{3}{2}$

The *y*-intercept is $\dfrac{3}{2}$.

Step 3: $3x-3=0$

$3x=3$

$x=1$

The *x*-intercept is 1.

Step 4: $x-2=0$

$x=2$

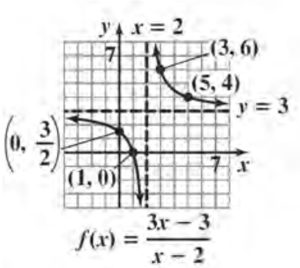
The line $x=2$ is the only vertical asymptote for the graph of *f*.

Step 5: The numerator and denominator have the same degree, 1. The leading coefficients of the numerator and denominator are 3 and 1, respectively. Thus, the equation of the horizontal asymptote is $y=\dfrac{3}{1}$ or $y=3$.

Step 6: Plot points between and beyond each *x*-intercept and vertical asymptote:

x	-1	$\frac{3}{2}$	3	5
$f(x)$	2	-3	6	4

Step 7: Use the preceding information to graph the function.



Pencil Problem #6

6a. Graph: $f(x)=\dfrac{-x}{x+1}$

6b. Graph: $f(x)=\dfrac{x^4}{x^2+2}$

Step 1: $f(-x)=\dfrac{(-x)^4}{(-x)^2+2}=\dfrac{x^4}{x^2+2}=f(x)$

Because $f(-x)=f(x)$, the graph has *y*-axis symmetry.

Step 2: $f(0)=\dfrac{0^4}{0^2+2}=0$

The *y*-intercept is 0, so the graph passes through the origin.

Step 3: $x^4=0$

$x=0$

There is only one *x*-intercept. This verifies that the graph passes through the origin.

Step 4: $x^2+2=0$

$x^2=-2$

$x=\pm i\sqrt{2}$

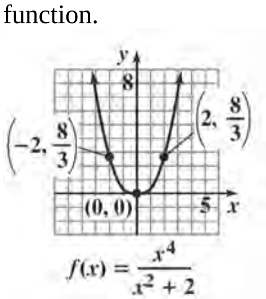
Since these solutions are not real, the graph of *f* will not have any vertical asymptotes.

Step 5: The degree of the numerator, 4, is greater than the degree of the denominator, 2, so the graph will not have a horizontal asymptote.

Step 6: Plot some points other than the intercepts:

x	-2	-1	1	2
$f(x)$	$\frac{8}{3}$	$\frac{1}{3}$	$\frac{8}{3}$	$\frac{1}{3}$

Step 7: Use the preceding information to graph the function.



6b. Graph: $f(x)=-\dfrac{1}{x^2-4}$

Objective #7: Identify slant asymptotes.

Solved Problem #7

7. Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$.

Note that the graph of *f* has a slant asymptote because the degree of the numerator is exactly one more than the degree of the denominator and the denominator is not a factor of the numerator.

Divide $2x^2 - 5x + 7$ by $x - 2$.

$$\begin{array}{r|rrr} 2 & 2 & -5 & 7 \\ & & 4 & -2 \\ \hline & 2 & -1 & 5 \end{array}$$

So, $\dfrac{2x^2 - 5x + 7}{x - 2} = 2x - 1 + \dfrac{5}{x - 2}$.

The equation of the slant asymptote is $y = 2x - 1$.

Pencil Problem #7

7. Find the slant asymptote of $f(x) = \dfrac{x^2 + x - 6}{x - 3}$.

Objective #8: Solve applied problems involving rational functions.

Solved Problem #8

8. A company is planning to manufacture wheelchairs. The cost, C, in dollars, of producing x wheelchairs is $C(x) = 500{,}000 + 400x$.

8a. Write the average cost function, $\bar{C}$.

The average cost is the cost divided by the number of wheelchairs produced.

$$\bar{C}(x) = \frac{500{,}000 + 400x}{x}$$

Pencil Problem #8

8. A company is planning to manufacture mountain bikes. The cost, C, in dollars, of producing x mountain bikes is $C(x) = 100{,}000 + 100x$.

8a. Write the average cost function, $\bar{C}$.

8b. Find and interpret $\bar{C}(1000)$ and $\bar{C}(10,000)$.

$$\bar{C}(1000) = \frac{500,000 + 400(1000)}{1000} = 900$$

The average cost per wheelchair of producing 1000 wheelchairs is $900.

$$\bar{C}(10,000) = \frac{500,000 + 400(10,000)}{10,000} = 405$$

The average cost per wheelchair of producing 10,000 wheelchairs is $405.

8c. What is the horizontal asymptote for the graph of $\bar{C}$? Describe what this means for the company.

The horizontal asymptote is $y = \frac{400}{1}$ or $y = 400$.

The cost per wheelchair approaches $400 as more wheelchairs are produced.

8b. Find and interpret $\bar{C}(1000)$ and $\bar{C}(4000)$.

8c. What is the horizontal asymptote for the graph of $\bar{C}$? Describe what this means for the company.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $\{x|x \neq -7, x \neq 7\}$ or $(-\infty, -7)\cup(-7, 7)\cup(7, \infty)$ *(3.5 #5)*

1b. all real numbers or $(-\infty, \infty)$ *(3.5 #7)*

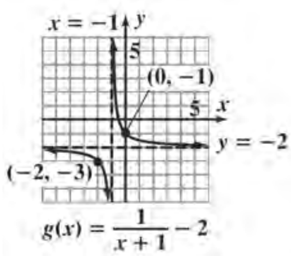
2. True *(3.5 #14)*

3a. vertical asymptote: $x = -4$ *(3.5 #25)*

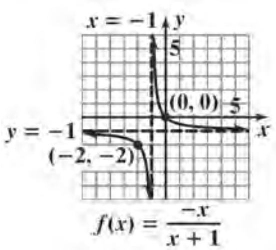
3b. no vertical asymptotes *(3.5 #27)*

4a. horizontal asymptote: $y = \frac{-2}{3}$ *(3.5 #43)*

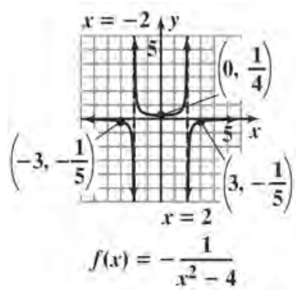
4b. horizontal asymptote: $y = 0$ *(3.5 #37)*

5. 

(3.5 #49)

6a. 

(3.5 #63)

6b. 

(3.5 #65)

7. $y = x + 4$ *(3.5 #85a)*

8a. $\bar{C}(x) = \frac{100{,}000 + 100x}{x}$ *(3.5 #99b)*

8b. $\bar{C}(1000) = 200$; The average cost per mountain bike of producing 1000 mountain bikes is \$200;
$\bar{C}(4000) = 125$; The average cost per mountain bike of producing 4000 mountain bikes is \$125. *(3.5 #99c)*

8c. $y = 100$; The cost per mountain bike approaches \$100 as more mountain bikes are produced. *(3.5 #99d)*

Section 3.6
Polynomial and Rational Inequalities

Tailgaters Beware!

It is never a good idea to follow too closely behind the car in front of you.
But when the roads are wet it can be even more dangerous.

In this section, we apply the mathematical concepts we learn to explore the different stopping distances required for a car driving on wet pavement and a car driving on dry pavement.

Objective #1: Solve polynomial inequalities.

✔ ***Solved Problem #1***

1. Solve and graph the solution set on a real number line: $x^2 - x > 20$

$$x^2 - x > 20$$
$$x^2 - x - 20 > 0$$

Solve the related quadratic equation to find the boundary points.

$$x^2 - x - 20 = 0$$
$$(x+4)(x-5) = 0$$

Apply the zero-product principle.

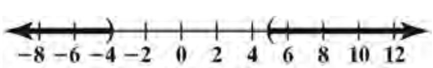
$x + 4 = 0$ or $x - 5 = 0$

$x = -4$ $\quad$ $x = 5$

The boundary points are -4 and 5.

Interval	Test Value	Test	Conclusion
$(-\infty, -4)$	-5	$(-5)^2 - (-5) > 20$ $30 > 20$, true	$(-\infty, -4)$ belongs to the solution set.
$(-4, 5)$	0	$(0)^2 - (0) > 20$ $0 > 20$, false	$(-4, 5)$ does not belong to the solution set.
$(5, \infty)$	10	$(10)^2 - (10) > 20$ $90 > 20$, true	$(5, \infty)$ belongs to the solution set.

The solution set is $(-\infty, -4) \cup (5, \infty)$.



Pencil Problem #1

1. Solve and graph the solution set on a real number line: $4x^2 + 7x < -3$

Objective #2: Solve rational inequalities.

✔ ***Solved Problem #2***

2. Solve and graph the solution set on a real number line: $\dfrac{2x}{x+1} \geq 1$

$$\frac{2x}{x+1} \geq 1$$
$$\frac{2x}{x+1} - 1 \geq 0$$
$$\frac{2x}{x+1} - \frac{x+1}{x+1} \geq 0$$
$$\frac{2x - x - 1}{x+1} \geq 0$$
$$\frac{x-1}{x+1} \geq 0$$

Find the values of *x* that make the numerator and denominator zero.

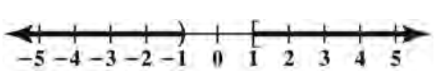
$x - 1 = 0$ and $x + 1 = 0$

$x = 1$ $\qquad x = -1$

The boundary points are -1 and 1.

Interval	Test Value	Test	Conclusion
$(-\infty, -1)$	-2	$\dfrac{2(-2)}{-2+1} \geq 1$ $4 \geq 1$, true	$(-\infty, -1)$ belongs to the solution set.
$(-1, 1)$	0	$\dfrac{2(0)}{0+1} \geq 1$ $0 \geq 1$, false	$(-1, 1)$ does not belong to the solution set.
$(1, \infty)$	2	$\dfrac{2(2)}{2+1} \geq 1$ $\dfrac{4}{3} \geq 1$, true	$(1, \infty)$ belongs to the solution set.

Exclude -1 from the solution set because it would make the denominator zero. The solution set is $(-\infty, -1) \cup [1, \infty)$.



Pencil Problem #2

2. Solve and graph the solution set on a real number line: $\dfrac{x+1}{x+3}<2$

Objective #3: Solve problems modeled by polynomial or rational inequalities.

✔ ***Solved Problem #3***

3. An object is propelled straight up from ground level with an initial velocity of 80 feet per second. Its height at time t is modeled by $s(t) = -16t^2 + 80t$ where the height, $s(t)$, is measured in feet and the time, t, is measured in seconds. In which time interval will the object be more than 64 feet above the ground?

To find when the object will be more than 64 feet above the ground, solve the inequality $-16t^2 + 80t > 64$.
Solve the related quadratic equation.

$$-16t^2 + 80t = 64$$
$$-16t^2 + 80t - 64 = 0$$
$$t^2 - 5t + 4 = 0$$
$$(t-4)(t-1) = 0$$
$$t - 4 = 0 \quad \text{or} \quad t - 1 = 0$$
$$t = 4 \qquad\qquad t = 1$$

The boundary points are 1 and 4.

Interval	Test Value	Test	Conclusion
$(0,1)$	0.5	$-16(0.5)^2 + 80(0.5) > 64$ $36 > 64$, false	$(0,1)$ does not belong to the solution set.
$(1,4)$	2	$-16(2)^2 + 80(2) > 64$ $96 > 64$, true	$(1,4)$ belongs to the solution set.
$(4,\infty)$	5	$-16(5)^2 + 80(5) > 64$ $0 > 64$, false	$(4,\infty)$ does not belong to the solution set.

The solution set is $(1,4)$.

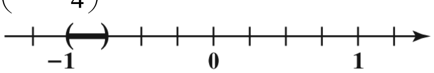
This means that the object will be more than 64 feet above the ground between 1 and 4 seconds excluding $t = 1$ and $t = 4$.

Pencil Problem #3

3. You throw a ball straight up from a rooftop 160 feet high with an initial speed of 48 feet per second. The function $s(t) = -16t^2 + 48t + 160$ models the ball's height above the ground, $s(t)$, in feet, t seconds after it was thrown. During which time period will the ball's height exceed that of the rooftop?

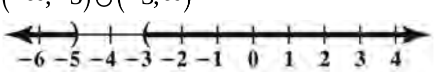
<u>Answers</u> <u>for Pencil Problems</u> <u>*(Textbook Exercise references in parentheses)*</u>:

1. $\left(-1, -\frac{3}{4}\right)$



(3.6 #15)

2. $(-\infty, -5) \cup (-3, \infty)$



(3.6 #55)

3. The ball exceeds the height of the building between 0 and 3 seconds. *(3.6 #76)*

Section 3.7
Modeling Using Variation

How Far Would You Go To Lose Weight?

On the moon your weight would be significantly less.

To find out how much less,
be sure to work on the application problems
in this section of your textbook!

Objective #1: Solve direct variation problems.

✔ *Solved Problem #1*

1. The number of gallons of water, W, used when taking a shower varies directly as the time, t, in minutes, in the shower. A shower lasting 5 minutes uses 30 gallons of water. How much water is used in a shower lasting 11 minutes?

Since W varies directly with t, we have $W = kt$.
Use the given values to find k.

$$W = kt$$
$$30 = k \cdot 5$$
$$\frac{30}{5} = \frac{k \cdot 5}{5}$$
$$6 = k$$

The equation becomes $W = 6t$. Find W when $t = 11$.

$$W = 6t$$
$$W = 6 \cdot 11$$
$$= 66$$

An 11 minute shower will use 66 gallons of water.

Pencil Problem #1

1. An alligator's tail length, T, varies directly as its body length, B. An alligator with a body length of 4 feet has a tail length of 3.6 feet. What is the tail length of an alligator whose body length is 6 feet?

Objective #2: Solve inverse variation problems.

✔ Solved Problem #2

2. The length of a violin string varies inversely as the frequency of its vibrations. A violin string 8 inches long vibrates at a frequency of 640 cycles per second. What is the frequency of a 10-inch string?

Beginning with $y = \frac{k}{x}$, we will use *l* for the length of the string and *f* for the frequency.

Use the given values to find *k*.

$$f = \frac{k}{l}$$
$$640 = \frac{k}{8}$$
$$8 \cdot 640 = 8 \cdot \frac{k}{8}$$
$$5120 = k$$

The equation becomes $f = \frac{k}{l}$

$$f = \frac{5120}{l}$$

Find *f* when $l = 10$.

$$f = \frac{5120}{l}$$
$$f = \frac{5120}{10}$$
$$f = 512$$

A string length of 10 inches will vibrate at 512 cycles per second.

Pencil Problem #2

2. A bicyclist tips his bicycle when making a turn. The angle *B*, formed by the vertical direction and the bicycle, is called the banking angle. The banking angle varies inversely as the cycle's turning radius. When the turning radius is 4 feet, the banking angle is $28°$. What is the banking angle when the turning radius is 3.5 feet?

Objective #3: Solve combined variation problems.

✔ *Solved Problem #3*

3. The number of minutes needed to solve an Exercise Set of variation problems varies directly as the number of problems and inversely as the number of people working to solve the problems. It takes 4 people 32 minutes to solve 16 problems. How many minutes will it take 8 people to solve 24 problems?

Let $m =$ the number of minutes needed to solve an exercise set.
Let $p =$ the number of people working on the problems.
Let $x =$ the number of problems in the exercise set.

Use $m = \frac{kx}{p}$ to find *k*.

$$m = \frac{kx}{p}$$
$$32 = \frac{k16}{4}$$
$$32 = 4k$$
$$k = 8$$

Thus, $m = \frac{8x}{p}$.

Find *m* when $p = 8$ and $x = 24$.

$$m = \frac{8 \cdot 24}{8}$$
$$m = 24$$

It will take 24 minutes for 8 people to solve 24 problems.

Pencil Problem #3

3. Body-mass index, or BMI, varies directly as one's weight, in pounds, and inversely as the square of one's height, in inches. A person who weighs 180 pounds and is 5 feet, or 60 inches, tall has a BMI of 35.15. What is the BMI, to the nearest tenth, for a 170 pound person who is 5 feet 10 inches tall?

Objective #4: Solve problems involving joint variation.

✔ *Solved Problem #4*

4. The volume of a cone, *V*, varies jointly as its height, *h*, and the square of its radius, *r*. A cone with a radius measuring 6 feet and a height measuring 10 feet has a volume of 120π cubic feet. Find the volume of a cone having a radius of 12 feet and a height of 2 feet.

Find *k*: $V = khr^2$

$$120\pi = k \cdot 10 \cdot 6^2$$
$$120\pi = k \cdot 360$$
$$\frac{120\pi}{360} = \frac{k \cdot 360}{360}$$
$$\frac{\pi}{3} = k$$

Thus, $V = \frac{\pi}{3}hr^2 = \frac{\pi hr^2}{3}$.

$$V = \frac{\pi hr^2}{3}$$
$$V = \frac{\pi \cdot 2 \cdot 12^2}{3} = 96\pi$$

The volume of a cone having a radius of 12 feet and a height of 2 feet is 96π cubic feet.

Pencil Problem #4

4. The heat loss of a glass window varies jointly as the window's area and the difference between the outside and inside temperatures. A window 3 feet wide by 6 feet long loses 1200 Btu per hour when the temperature outside is $20°$ colder than the temperature inside. Find the heat loss through a glass window that is 6 feet wide by 9 feet long when the temperature outside is $10°$ colder than the temperature inside.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. 5.4 feet *(3.7 #21)* **2.** $32°$ *(3.7 #27)* **3.** BMI: 24.4 *(3.7 #31)* **4.** 1800 Btu *(3.7 #33)*