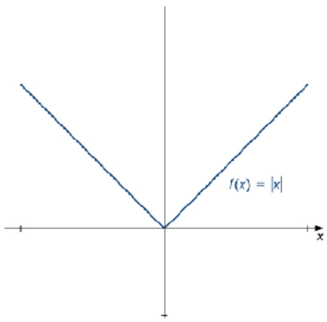
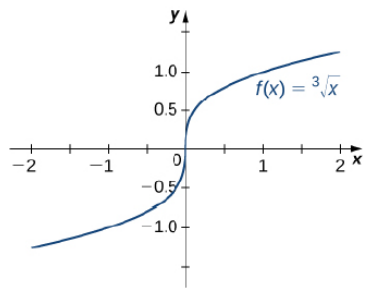
MATH 141 OpenStax Name___________________________

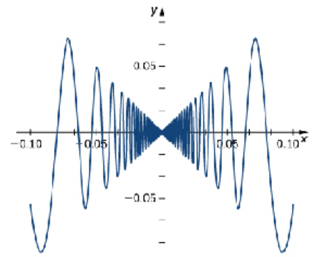
Section 3.2 The Derivative as a function

Theorem Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.







The function $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x\neq 0 \\ 0 \text{ if } x=0 \end{cases}$ is not differentiable at 0.

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x)=|x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.
3. As we saw in the example of $f(x)=\sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x\neq 0 \\ 0 \text{ if } x=0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

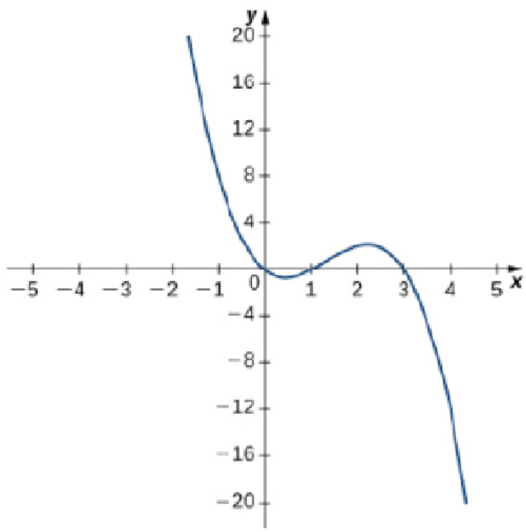
$$f''(x),\ f'''(x),\ f^{(4)}(x),\ldots,\ f^{(n)}(x)$$

$$y''(x),\ y'''(x),\ y^{(4)}(x),\ldots,\ y^{(n)}(x)$$

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dy^3},\ \frac{d^4y}{dy^4},\ldots,\ \frac{d^ny}{dy^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}.$

For the following exercises, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.



For the following exercises, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

$$\lim_{h \to 0} \frac{(1+h)^{2/3} - 1}{h}$$

$$\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$$

$$\lim_{h \to 0} \frac{[2(3+h)^2 - (3+h)] - 15}{h}$$

For the following function

a. sketch the graph and

b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

$$f(x) = \begin{cases} 2\sqrt{x}, & 0 \le x \le 1 \\ 3x - 1, & x > 1 \end{cases}$$